Proceedings of the Jangjeon Mathematical Society
20 (2017), No. 3. pp. 347 – 351

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2017.20.3.347

COMPLETE LEFT IDEALS IN TERNARY SEMIGROUPS

BINDU.P, SARALA.Y, SRINIVASA RAO.T, MADHUSUDHANA RAO.D

Abstract. In this paper we introduce the notions of complete leftideals in ternarysemigroups. We also define direct product and left unit of a ternary-semigroups. And obtain some results from these ideals. We obtain the relation between the complete leftideals and set of all $(0, 1)$ regular elements of a ternarysemigroups.
MSC(2010): Primary: 20M12; Secondary: 20M17.

1. Introduction

In 1961, the theory of semigroups was investigated by clifford[2]. In 1962, Petrich[5] first introduced the concept of prime ideals of the cartesian product of two semigroups and proved some results. We refer several prime ideal theorems in ternarysemigroups was investigated by Shabbir[7] and also different types of ideal theorems in ternary semigroups (see [4,6,8]).

Our aim in this paper is to develop a body of results on the complete leftideals of ternarysemigroups, that can be used like the more classical results on ternary-semigroups. A pioneering and inspiring work in this direction is briefly sketched at the paper by Fabrici[3], but it long looked like an isolated attempt.

Definition 2.1: A nonempty subset L of a ternarysemigroup T is a complete leftideal of T if $TTL = L$.

A nonempty subset L of a ternarysemigroup T is said to be a complete leftideal of T if $a \in L$ there exist $x, y \in T$; $b \in L$ such that $xyb = a$.

Theorem 2.2: The non empty union of any two complete leftideals of a ternary-semigroup T is a complete leftideal of T.

Proof: Let L_1, L_2 be two complete leftideals of T. Then $TTL_1 = L_1$; $TTL_2 = L_2$.

Hence $TT(L_1 \cup L_2) = TTL_1 \cup TTL_2 = L_1 \cup L_2$. Which proves our assertion. The next example gives a negative answer.

Example 2.3: Let $T = \{w, x, y, z\}$ be a ternarysemigroup with the multiplication table.

	w	x	y	z
w	w	w	w	w
x	w	w	x	x
y	w	w	y	z
z	w	w	y	z

$L_1 = \{w, x, y\}$, $L_2 = \{w, x, z\}$ are complete leftideals in T, but $L_1 \cap L_2 = L_3 = \{w, x\}$ is not a complete leftideal of T.

Hence the intersection of two complete leftideals is not a complete leftideal.

Key words and phrases. Complete ieft ideals, left unit, Direct product of ternarysemigroups.

Theorem 2.4: The union of any family of complete leftideals of a ternarysemigroup T is a complete leftideal of T.

Proof: Let $\{L_\alpha\}_{\alpha\in\Delta}$ be a family of complete leftideals of a ternarysemigroup T. Let $L = \bigcup_{\alpha\in\Delta} L_\alpha$. Clearly L is a non empty subset of T. Let $a \in L$; $x, y \in T$

$a \in L \Rightarrow a \in \bigcup_{\alpha\in\Delta} L_\alpha \Rightarrow a \in L_\alpha$ for some $\alpha \in \Delta$. $a \in L_\alpha \ni x, y \in T$, $b \in L_\alpha$; L_α is a leftideal of $T \Rightarrow xyb = a \in \bigcup_{\alpha\in\Delta} L_\alpha = L \Rightarrow xyb = a$. Therefore L is a complete leftideal of T.

Definition 2.5: A leftideal L of a ternarysemigroup T is called a minimal if there exist no leftideal of T properly contained in L. Evidently, every minimal leftideal of a ternarysemigroup T is a complete leftideals of T.

Definition 2.6: An element a of a ternarysemigroup T satisfies the condition (m, n) if there exist an element $x \in T$ such that $a = a^m x a^n$. Here m,n are non-negitive integers, a^0 means the void symbol. The set of all elements satisfying the condition (m, n) is called a class of regularity and will be denoted by $R_t\,(m, n)$.

If all the elements of a ternarysemigroup T satisfy the condition (m, n) we shall write $T = R_t\,(m, n)$.

Theorem 2.7: Every leftideal of a ternarysemigroup T is a complete leftideal of T if and only if $T = R_t\,(0, 1)$.

Proof: Suppose every leftideal of T be complete. Let a be any element of T. The left ideal $a \cup TTa$ satisfies

$TT\,(a \cup TTa) = a \cup TTa$

i.e $TTa \cup TTTTa = a \cup TTa$.

Hence $TTa = a \cup TTa$.

Therefore $a \in TTa$, which proves that $T = R_t\,(0, 1)$.

Conversely assume that $T = R_t\,(0, 1)$.

Let $L = \bigcup_{a\in L} a$ be a left ideal of T.

Then $TTL \supseteq \left\{\bigcup_{a\in L} x_a\right\}\left\{\bigcup_{a\in L} x_a\right\}\left\{\bigcup_{a\in L} a\right\} \supseteq \bigcup_{a\in L} x_a x_a a = \bigcup_{a\in L} a = L$.
On the other hand since L is a left ideal , $TTL \subseteq L$. Hence $TTL = L$. Hence L is a complete leftideal.

Corollary 2.8: Every leftideal of a ternarysemigroup T is a complete leftideal of T if and only if $T = R_t\,(0, 1) = R_t\,(1, 0)$.

Definition 2.9: An element a of a ternarysemigroup T is said to be a left unit of T provided $aat = t \;\forall t \in T$.

Theorem 2.10: If T contains a left unit , then every leftideal is complete.

Proof: Let t be a left unit of T and L be a leftideal of T. Let L be a leftideal of $T \Rightarrow TTL \subseteq L$. Let $x \in L \Rightarrow x \in T$ and t is a left unit of T.

$\Rightarrow x = ttx \in TTL$. Hence $L \subseteq TTL$. Therefore $L = TTL$ and hence L is a complete leftideal of T.

Theorem 2.11: If all leftideals of T are complete, then $T^3 = T$.

Proof: Suppose all leftideal of T are complete, then T itself is the leftideal of T. Hence $T^3 = T$.

Theorem 2.12: If $T = R_t\,(1, 1)$, then every leftideal of T is complete.

Proof: Similar to therorem 2.7.

The next example of a ternarysemigroup shows that the converse of the theorem 2.11 need not hold.

Example 2.13: Let T be an additive ternarysemigroup of positive numbers. Then $T^3 = T$. Let $\langle a, \infty)$ with $a > 0$. Then $TTL = (a, \infty) \subset \langle a, \infty)$ so that L is not complete.

If not every leftideal of a ternarysemigroup T is complete, then essentially less can be said about the ternarysemigroup. This statement holds.

Theorem 2.14: If $L \subseteq R_t(0, 1)$, where L is a leftideal of a ternarysemigroup T, then L is complete leftideal of T.

Proof: The statement follows from the assumption and from part of the proof of theorem 2.7.

The ternarysemigroup in example 2.13 shows that the converse is not true. It is sufficient to take $L = (a, \infty)$. It can be easily shown that L is complete but $L \subset R_t(0, 1)$ does not hold.

Definition 2.15: Let $\{T_i\}_{i \in I}$ be an arbitrary system of ternarysemigroups. Denote by T the set of all functions ξ defined on I such that $\xi(i) \in T_i$. Introduce in T a multiplication in this way:

If $x, y, z \in T$ are arbitrary elements of T. Then the product $p = xyz$ is given by $p(i) = x(i)\, y(i)\, z(i)$ for every $i \in I$. The set T with this multiplication is a ternarysemigroup, which is called a direct product of ternarysemigroups $\{T_i\}$, $i \in I$ and is denoted by $T = \prod_{i \in I} T_i$.

Theorem 2.16: If L_i is a leftideal of a ternarysemigroup $\{T_i\}$, $i \in I$, then $L = \prod_{i \in I} L_i$ is a leftideal of the ternarysemigroup $T = \prod_{i \in I} T_i$.

Proof: Let $a \in L$, $b, c \in T$;
$a \in L \Rightarrow a \in \prod_{i \in I} L_i$
$\Rightarrow a \in L_i$ for each $i \in I$. $a \in L_i$; $b, c \in T_i$; L_i is a leftideal of $T_i \Rightarrow bca \in L_i$.
$bca \in L_i$, for all $i \in I \Rightarrow bca \in \prod_{i \in I} L_i \Rightarrow bca \in L$.

This proves our statement.

Let us put the question, whether the completeness of leftideals L_i in T_i, $i \in I$, implies the completeness of leftideal $L = \prod_{i \in I} L_i$ in $T = \prod_{i \in I} T_i$.

Theorem 2.17: Let L_i for every $i \in I$ a complete leftideal of the ternarysemigroup T_i, then $L = \prod_{i \in I} L_i$ is a complete leftideal of $T = \prod_{i \in I} T_i$.

Proof: Let L_i be a complete leftideals of a ternary semigroup T_i, hence $T_i T_i L_i = L_i$.

We have to prove that for any $\mu \in L$, there exist $\nu \in L$ and $\alpha, \beta \in T$ such that $\alpha\beta\nu = \mu$

Since L_i is a complete leftideal of T_i, there exist for every $\mu(i) = a_i \in L_i$ three elements $b_i \in L_i$ and $x_i, y_i \in T_i$ such that $x_i y_i b_i = a_i$. The function ν, α, β defined by $\nu(i) = b_i$; $\alpha(i) = x_i$; $\beta(i) = y_i$ satisfy $\alpha\beta\nu = \mu$.

This proves our statement.

Theorem 2.18: Let L be a complete leftideal of a ternarysemigroup $T = \prod_{i \in I} T_i$. Then

a) $P_i(L)$ is a complete leftideal of T_i.
b) $\prod_{i \in I} P_i(L)$ is a complete leftideal of T_i.

Proof: a) Let L be a complete leftideal of a ternarysemigroup $T = \prod_{i \in I} T_i$. The fact that the $P_i(L)$ is a leftideal of T_i is known from [3]. It is only necessary to prove that it is complete. Let $\alpha(i) \in P_i(L)$. To prove that $P_i(L)$ is a complete leftideal, it is sufficient to show that there exist $b_i \in P_i(L)$ and $x_i, y_i \in T_i$ such that $x_i y_i b_i = a_i$.

Since $a_i \in P_i(L)$, it follows that there exists an element $\mu \in L$ such that $\mu(i) = a_i$. Since L is a complete leftideal of $T = \prod_{i \in I} T_i$, there exists an element $\nu \in L$ and $\alpha, \beta \in T$ such that $\alpha\beta\nu = \mu$.

This means that for every $i \in I$, we have $\alpha(i)\beta(i)\nu(i) = \mu(i)$ where $\mu(i) = a_i$, $\nu(i) = b_i \in P_i(L)$ and $\alpha(i) = x_i \in T_i$; $\beta(i) = y_i \in T_i$. Therefore, we have $x_i y_i b_i = a_i$. This proves (a).

b) The statement (b) follows from (a) and theorem 2.17.

Theorem 2.19: A ternarysemigroup $T = \prod_{i \in I} T_i$ satisfies the condition (m, n) if and only if each of the ternarysemigroup T_i satisfies the condition.

Proof: Let us assume that every ternarysemigroup T_i satisfies condition (m, n). Let $\alpha \in T$ be an arbitrary element. Then $\alpha(i) = a_i \in T_i$ for every $i \in I$.

Since T_i satisfies condition (m, n) there exist $x_i \in T_i$
such that $a_i = a_i^m x_i a_i^n \rightarrow (1)$

Define $\eta \in T$ by the requirement that $\eta(i) = x_i$ for every $i \in I$. The relation (1) can be written in the form $\alpha(i) = [\alpha(i)]^m \eta(i) [\alpha(i)]^n$ for every $i \in I$. This means $\alpha = \alpha^m \eta \alpha^n$. But the last relation says that $T = \prod_{i \in I} T_i$ satisfies the condition (m, n).

Let $T = \prod_{i \in I} T_i$ satisfies the condition (m, n). Let $a_i \in T_i$ be an arbitrary element. Then there exist at least one element $\alpha \in T$ such that $\alpha(i) = a_i$. Since T satisfies the condition (m, n), there exist an element $\eta \in T$ such that $\alpha = \alpha^m \eta \alpha^n$. Hence for our i, $a_i = a_i^m x_i a_i^n$. This means that T_i satisfies the condition (m, n).

Theorem 2.7 and 2.19 imply:

Corollary 2.20: Every leftideal of the ternary semigroup $T = \prod_{i \in I} T_i$ is complete if and only if every leftideal of the ternarysemigroup $T_i\,(i \in I)$ is complete.

Corollary 2.21: The following statements are equivalent.

a) Each of the ternarysemigroup $T_i\,(i \in I)$ satisfy condition $(0, 1)$.
b) The ternarysemigroup $T = \prod_{i \in I} T_i$ satisfy condition $(0, 1)$.
c) Every leftideal $T_i\,(i \in I)$ is complete.
d) Every leftideal of $T = \prod_{i \in I} T_i$ is complete.

Proof: (a) $\Rightarrow$ (b) according to theorem 2.19.
(b) $\Rightarrow$ (c) according to theorem 2.19 and theorem 2.7.
(c) $\Rightarrow$ (d) according to corollary 2.20.
(d) $\Rightarrow$ (a) according to corollary 2.20 and theorem 2.7.

Definition 2.22: A leftideal L of a ternarysemigroup T is called a semiprime if for every element $a \in T$ and an arbitrary integer n the relation $a^n \in L$ implies $a \in L$.

Theorem 2.23: Every leftideal of a ternarysemigroup T is a semiprime ideal if and only if T satisfies condition $(0, 2)$

Proof: Suppose that L is a leftideal of a ternarysemigroup T and $l \in L$.
$l \in L$; L is semiprime $\Rightarrow \langle l \rangle^3 \subseteq L \Rightarrow l \in L$. If $\langle l \rangle^3 \subseteq L \Rightarrow \langle l^3 \rangle \subseteq L \Rightarrow l^3 \in L$.

we can write $l = l^3 = l^0 l l^2$.
Therefore T satisfies the condition $(0, 2)$.

Conversely suppose that T satisfies the condition $(0, 2)$ and L is a leftideal of T. Then there exist $a \in T$ such that $a = a^0 a a^2 \Rightarrow a = a^3 \Rightarrow a^3 \in T. \Rightarrow \langle a^3 \rangle \subseteq L \Rightarrow \langle a \rangle^3 \subseteq L$. Therefore $\langle a \rangle^3 \subseteq L \Rightarrow a \in L \Rightarrow L$ is semiprime.

Theorem 2.24: Let L_i be a left semiprime ideal T_i for every $i \in I$. Then $L = \prod_{i \in I} L_i$ is a left semiprime ideal of $T = \prod_{i \in I} T_i$.

Proof: Let $\alpha \in T = \prod_{i \in I} T_i$ be an arbitrary element and let $\alpha^n \in L = \prod_{i \in I} L_i$. Then $[\alpha(i)]^n \in L_i$ for every $i \in I$.

Since L_i is a semiprime ideal of T_i, we have $\alpha(i) \in L_i$ for every $i \in I$. Hence $\alpha \in L = \prod_{i \in I} L_i$.

Corollary 2.24: Let every leftideal of a ternarysemigroup T_i be a semiprime ideal of T_i for every $i \in I$. Then

a) Every leftideal of $T = \prod_{i \in I} T_i$ is a semiprime ideal of T.

b) Every leftideal of $T = \prod_{i \in I} T_i$ is a complete leftideal of T.

Proof: The statement (a) follows from theorem 2.19 and theorems 2.23 and 2.24.

The statement (b) follows from the relation $R_t(m_1, n_1) \subseteq R_t(m_2, n_2)$ If $m_1 \geq m_2$; $n_1 \geq n_2$ and from theorem 2.19.

References

[1] Anjaneyulu.A, *Structure and ideal theory of semigroups,* Thesis,ANU (1980).

[2] Clifford.A.H, Preston.G.B *The algebraic theory of semigroups,* vol.1, American Math.society,providence,(1961)

[3] Fabrici.I, *On complete ideals in semigroups,* Matematicky casopis., **18**(1968), No.1,34-39.

[4] Kar.S, *On ideals in ternary semigroups,* Int.Jour.Math.Gen.Sci., **18**(2005), 3015-3023.

[5] Petrich.M, *Prime ideals of the cartesian product of two semigroups,* Czechoslovak mathematical journal., **12**(1962),No.1, 150-152.

[6] Sarala.y, Anjaneyulu.A and Madhusudhana Rao.D, *Ideals in ternary semigroups,* International e-journal of mathematics and engineering., **203**(2013), 1950-1968.

[7] Shabir.M and Bashir.S *Prime ideals in ternary semigroups,* Asian-European journal of mathematics., **2**(2009), 139-152.

[8] Sioson.F.M, *Ideal theory in ternary semigroups,* Math.Japan, **10**(1965), 63-84.

Bindu.P,Department of Mathematics, K.L.University Guntur Dt., A.P., INDIA.
E-mail address: bindunadndla@yahoo.com

Dr.sarala.yella;M.sc,M.phill,ph.D; Department of Mathematics,, National Institute of Technology A.P., INDIA.
E-mail address: saralayella1970@gmail.com

Srinivasa Rao.T; M.sc,M.phill,ph.D; Department of Mathematics,, K.L.University Guntur Dt., A.P., INDIA.